A note on the exact lattice chiral symmetry in the overlap formalism

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Abstract

Using the grassman-number-integral representation of the vacuum overlap formula, it is shown that the symmetry of the auxiliary quantum fermion system in the overlap formalism induces exact chiral symmetry of the action of the type given by Luscher under the chiral transformation $\delta\psi_n = \gamma_5(1 - aD)\psi_n$ and $\delta\bar\psi_n = \bar\psi_n\gamma_5$. With this relation, we consider the connection between the covariant form of the anomaly discussed in the context of the overlap formula and the axial anomaly associated to the exact chiral symmetry in the action formalism. The covariant gauge current in the overlap formalism is translated to the action formalism and its explicit expression is obtained.

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1 Introduction

The vacuum overlap formula[1, 2] provides a well-defined lattice regularization of the chiral determinant. It can reproduce the known features of the chiral determinant in the continuum theory: the one-loop effective action of the background gauge field[3] including the consistent anomaly[4], topological charges and fermionic zero modes associated with the topologically non-trivial gauge fields[1, 2, 5], the $SU(2)$ global anomaly[6, 7], and so on. These properties of this formalism allow the description of the fermion number violation on the lattice. Several numerical applications[8] have been performed. Their results strongly suggest that the overlap formalism can actually be a promising building block for the construction of lattice chiral gauge theories.

In this formalism, an auxiliary quantum Dirac fermion system plays a fundamental role:

$$\left\{ \hat{a}_{n\alpha}, \hat{a}^\dagger_{m\beta} \right\} = \delta_{nm}\delta_{\alpha\beta}, \quad (n, m \in Z^4 \ \alpha, \beta = 1, 2, 3, 4). \tag{1.1}$$

It is described by the two Hamiltonians

$$\mathcal{H} = \sum_{nm} \hat{a}^\dagger_n H_{nm} \hat{a}_m, \qquad \mathcal{H}_5 = \sum_n \hat{a}^\dagger_n \gamma_5 \hat{a}_n, \tag{1.2}$$

where H is the hermitian Wilson-Dirac operator defined by

$$H = \gamma_5 \left\{ \gamma_\mu \frac{1}{2} \left(\nabla_\mu - \nabla^\dagger_\mu \right) + \frac{a}{2} \nabla_\mu \nabla^\dagger_\mu - \frac{1}{a} m_0 \right\}, \quad (0 < m_0 < 2). \tag{1.3}$$

Here a certain compact gauge group G is assumed and ∇_μ is the gauge covariant forward difference operator. From the two vacua (Dirac seas) of these Hamiltonians, the chiral determinant is defined as

$$\det C \equiv \langle v \,|\, L \,\rangle. \tag{1.4}$$

Recently, it has been realized that this formalism can provide a solution of the Ginsparg-Wilson relation in the presence of gauge fields. Neuberger has proposed a Dirac operator which describes exactly massless fermions on the lattice [9, 10] based on the overlap formalism of chiral determinant [11, 12]. Its explicit form is known as

$$aD_{nm} = \left(1 + \gamma_5 \frac{H}{\sqrt{H^2}} \right)_{nm}, \tag{1.5}$$

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This overlap Dirac operator satisfies the Ginsprag-Wilson relation [13, 14, 10] [1]

$$D_{nm}\gamma_5 + \gamma_5 D_{nm} = \sum_l a D_{nl}\gamma_5 D_{lm}. \tag{1.6}$$

This relation guarantees that the effects of the chiral symmetry breaking terms in the Dirac operator appear only in local terms. This is the clue to escaping the Nielsen-Ninomiya theorem [19].

The locality properties of the overlap Dirac operator has been examined in detail by Hernandes, Jansen and Lüscher [25]. They have given a proof of the locality for a certain set of bounded small gauge fields and also for the case with an isolated zero mode of H. The locality in dynamically generated gauge fields at strong coupling has also been examined. Their data has provided a numerical evidence that the overlap Dirac operator is local with the gauge fields for $\beta \geq 6.0$ in $SU(3)$ gauge theory.

Furthermore, Lüscher pointed out that the Ginsparg-Wilson relation implies exact symmetry of the fermion action [21],

$$S_F = a^4 \sum_{nm} \bar{\psi}_n D_{nm}\psi_m. \tag{1.7}$$

The chiral transformation proposed is given as

$$\delta\psi_n = \sum_m \gamma_5 \left(1 - \frac{a}{2}D\right)_{nm}\psi_m, \quad \delta\bar{\psi}_n = \sum_m \bar{\psi}_m \left(1 - \frac{a}{2}D\right)_{mn}\gamma_5. \tag{1.8}$$

Lüscher also observed that for the flavor-singlet chiral transformation the functional integral measure is not invariant

$$\delta[d\psi d\bar{\psi}] = [d\psi d\bar{\psi}]\, a\mathrm{Tr}\left(\gamma_5 D\right). \tag{1.9}$$

This global anomalous variation is indeed given in terms of the index of the Dirac operator [17, 21].

$$- a\mathrm{Tr}\left(\gamma_5 D\right) = 2N_f\, \mathrm{index}\left(D\right). \tag{1.10}$$

Its local anomalous variation has been evaluated by the authors in the weak coupling expansion of the overlap Dirac operator [22], supplementing the previous calculation [13]. It has been shown to have the correct form of the topological charge density in the classical continuum limit:

$$\lim_{a\to 0}\left(-a\sum_n \alpha_n \mathrm{tr}\gamma_5 D_{nn}\right) = \frac{g^2 N_f}{32\pi^2}\int d^4x\, \alpha(x)\, \epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}^a(x)F_{\rho\sigma}^a(x). \tag{1.11}$$

[1] Another Dirac operator which satisfies the Ginsparg-Wilson relation has been proposed by Hasenfratz et. al. [14, 15, 16, 17, 18]

For abelian gauge theories, Lüscher has shown that the local anomalous variation can be written exactly by the topological charge density at finite lattice spacing up to total divergence [23].

It was Narayanan who pointed out that there is a close relation between the Ginsparg-Wilson relation and the idea of the overlap [24]. If we write the Dirac operator in the form as

$$D = 1 - \gamma_5 \, \epsilon, \tag{1.12}$$

the Ginsparg-Wilson relation reduces to a simple relation

$$\epsilon^2 = 1. \tag{1.13}$$

This means that ϵ can consist a projection operator and the Dirac operator can be written as a combination of this projector and the chiral projector:

$$D = 2 \left\{ P_L \left(\frac{1+\epsilon}{2} \right) + P_R \left(\frac{1-\epsilon}{2} \right) \right\}. \tag{1.14}$$

It follows from this that the determinant of D is factorized into two contributions from the eigenstates of ϵ (or γ_5) with opposite eigenvalues. The each factor can be regarded as the chiral determinant as an overlap of two vacua defined with Hamiltonians ϵ and γ_5. This chiral factorization which follows from the Ginsparg-Wilson relation was also discussed by Neidermeyer, Hasenfratz and Lüscher [29] in more general context where only the Ginsparg-Wilson relation and locality of the Dirac operator are assumed [29].

Moreover, as discussed by Narayanan and Neuberger [11], the quantum fermion system which underlies the overlap formalism possesses fermion symmetry. It is suggested by Neuberger in [27] that the exact chiral symmetry of Lüscher can be related to this symmetry of the quantum fermion system. It has also been discussed, by Narayanan and Neuberger [11], by Randjbar-Daemi and Strathdee [30] and by Neuberger[27, 28], how to obtain the covariant form of anomaly in the context of the overlap formalism.

In this paper, trying to understand the connection among these discussions, we will discuss what structure of the overlap formalism leads to the Ginsparg-Wilson relation and the associated exact chiral symmetry on the lattice. Through the grassman-number-integral representation of the overlap, we first establish an exploit relation between the canonical variables in the overlap formalism and the Dirac field variables in its action formalism. Then we will see that the symmetry of the quantum fermion system induces

exact chiral symmetry of the action under the chiral transformation defined by

$$\delta\psi_n = \gamma_5(1 - aD)\psi_n, \quad \delta\bar{\psi}_n = \bar{\psi}_n\gamma_5. \tag{1.15}$$

Note that this variant of Lüscher's chiral transformation was considered by Neidermeyer, Hasenfratz and Lüscher in more general context where only the Ginsparg-Wilson relation and locality of the Dirac operator are assumed [29]. Using this relation, we discuss the connection between the axial anomaly $-a\text{tr}\gamma_5 D_{nn}$ in the action formulation and the anomaly in the covariant form discussed by Narayanan and Neuberger [11], by Randjbar-Daemi and Strathdee [30] and by Neuberger[27, 28] in the context of the overlap formula.

2 Overlap in terms of grassman number integral

We start from rewriting the overlap formula of the chiral determinant in terms of the functional integral over the grassman number representatives of the canonical variables Eq. (1.1). With the canonical variables, the Fock vacuum is defined as

$$\hat{a}_{n\alpha}|0\rangle = 0, \tag{2.1}$$

and the two Hamiltonians are defined by Eqs. (1.2).

Let us denote the eigenvectors of H as $u_n(\lambda)$ for positive eigenvalues ($\lambda > 0$) and $v_n(\lambda)$ for negative eigenvalues ($\lambda < 0$). We denote by n both lattice and spinor indices. When we need to specify the chirality of the spinor variables, we use the abbreviations (nL) and (nR) for right-handed and left-handed, respectively. Note that it is possible for all gauge fields to fix the overall phase of eigenvectors by the condition

$$\det_{n,(\lambda>0,\lambda'<0)} \big(u_n(\lambda), \cdots, v_n(\lambda'), \cdots\big) = 1. \tag{2.2}$$

The diagonal basis of $\hat{H}$ can be written as

$$\hat{a}_u(\lambda) \;\; = \;\; \sum_n u_n^\dagger(\lambda)\hat{a}_n, \qquad \hat{a}_v(\lambda) = \sum_n v_n^\dagger(\lambda)\hat{a}_n. \tag{2.3}$$

$\hat{H}_5$ is diagonal in the chiral basis defined by

$$\hat{a}_{nL} = \left(\frac{1 - \gamma_5}{2}\right)\hat{a}_n, \qquad \hat{a}_{nR} = \left(\frac{1 + \gamma_5}{2}\right)\hat{a}_n. \tag{2.4}$$

Then the vacua of the Hamiltonians (Dirac seas) can be obtained as follows:

$$| v \rangle = \prod_{\lambda<0} a_v^\dagger(\lambda)|0\rangle, \quad | L \rangle = \prod_{nL} \hat{a}_{nL}^\dagger|0\rangle. \tag{2.5}$$

Here the subscript (nL) of the product stands for the product *over all the left-handed components* of a given field variable. The overlap formula of the chiral determinant is now defined by the inner product of these two Dirac vacua: [2]

$$\langle v \,|\, L \rangle = \det_{nL;\lambda<0} \left\{ v_{nL}^\dagger(\lambda) \right\}. \tag{2.11}$$

This inner product of two vacua may be written in terms of the functional integral over the grassman number representatives of the canonical variables [31, 32, 33]. In the grassman number representation (coherent state representation), the two Dirac vacua are represented by their wave functions

$$| v \rangle = \prod_{\lambda<0} \hat{a}_v^\dagger(\lambda)|0\rangle \quad \longrightarrow \quad \Psi_v(a_n^\dagger) = \prod_{\lambda<0} \left(\sum_n a_n^\dagger v_n(\lambda) \right), \tag{2.12}$$

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[2] We need to fix the phase of the vacuum $| v \rangle$. In the overlap formalism, it is fixed by refering to the free theory vacuum, following the Wigner-Brillouin phase convention [1]. That is, we assume that the eigenvectors satisfies

$$\langle v^0 \,|\, v \rangle = \det_{\lambda,\lambda'<0} \left\{ v_n^{0\dagger}(\lambda)v_n(\lambda') \right\} = \text{real positive}, \tag{2.6}$$

$$\langle u^0 \,|\, u \rangle = \det_{\lambda,\lambda'>0} \left\{ u_n^{0\dagger}(\lambda)u_n(\lambda') \right\} = \text{real positive}. \tag{2.7}$$

Note that this conditions are consistent with Eq. (2.2) [1]. This is easily seen from the following identity:

$$\det_{n:(\lambda>0,\lambda'<0)} \left(u_n(\lambda), v_n(\lambda') \right) = \det_{n:(\lambda^0>0,\lambda^{0'}<0)} \left(u_n^0(\lambda), v_n^0(\lambda') \right)^* \times \det_{n:(\lambda>0,\lambda'<0)} \left(u_n(\lambda), v_n(\lambda') \right)$$

$$= \det_{(\lambda^0>0,\lambda^{0'}<0),(\lambda>0,\lambda'<0)} \begin{pmatrix} u_n^{0\dagger}(\lambda^0)u_n(\lambda) & u_n^{0\dagger}(\lambda^0)v_n(\lambda') \\ v_n^{0\dagger}(\lambda^{0'})u_n(\lambda) & v_n^{0\dagger}(\lambda^{0'})v_n(\lambda') \end{pmatrix}$$

$$= \frac{\det_{\lambda^0,\lambda>0} \left(u_n^{0\dagger}(\lambda^0)u_n(\lambda) \right)}{\det_{\lambda^{0'},\lambda'<0} \left(v_n^{0\dagger}(\lambda^{0'})v_n(\lambda') \right)} > 0. \tag{2.8}$$

In the last line, we have used an identity concering the the determinant of a unitary matrix [1]: if an $N \times N$ unitary matrix U is written in block-wise with an $K \times K(0 < K < N)$ matrix A as

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.9}$$

its determinant can be evaluated as

$$\det U = \frac{\det A}{\det D^\dagger}, \tag{2.10}$$

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$$| L \rangle = \prod_{nL} \hat{a}_{nL}^{\dagger} |0\rangle \quad \longrightarrow \quad \Psi_L(a_n^{\dagger}) = \prod_{nL} a_{nL}^{\dagger}. \tag{2.13}$$

The inner product is then written by grassman number integral with the measure factor $\exp\left(-\sum_n a_n^{\dagger} a_n\right)$ as follows:

$$\begin{aligned}
\langle v \,|\, L \rangle &= \int \prod_n da_n da_n^{\dagger} \, \exp\left(-\sum_n a_n^{\dagger} a_n\right) \Psi_v^{\dagger}(a_n) \Psi_L(a_n^{\dagger}) \\
&= \int \prod_n da_n da_n^{\dagger} \, \exp\left(-\sum_n a_n^{\dagger} a_n\right) \prod_{\lambda<0} \left(\sum_n v_n(\lambda)^{\dagger} a_n\right) \prod_{n,L} a_{nL}^{\dagger}.
\end{aligned} \tag{2.14}$$

It is easy to check that the direct evaluation of the integral reproduces Eq. (2.11).

As easily seen in Eq. (2.14), the grassman integrals for the variables $a_{nL}^{\dagger}$ and $a_v(\lambda) = \sum_n v_n^{\dagger}(\lambda) a_n$ are all saturated by the vacuum wave functions. Then we may integrate out them all. This acts as the projection to the remaining variables $a_{nR}^{\dagger}$ and $a_u(\lambda) = \sum_n u_n^{\dagger}(\lambda) a_n$:

$$\prod da_n \Psi_v^{\dagger}(a_n) \, a_m = \prod da_n \Psi_v^{\dagger}(a_n) \left(\sum_{\lambda} u_n(\lambda) u_m^{\dagger}(\lambda)\right) a_m, \tag{2.15}$$

$$\prod d\bar{a}_n \Psi_L(a_n^{\dagger}) \, a_n^{\dagger} = \prod d\bar{a}_n \Psi_L(a_n^{\dagger}) \, a_n^{\dagger} P_R. \tag{2.16}$$

Accordingly, the overlap Eq. (2.14) may be written as

$$\begin{aligned}
\langle v \,|\, L \rangle &= \int \prod_n da_n da_n^{\dagger} \prod_{\lambda<0} \left(\sum_n v_n(\lambda)^{\dagger} a_n\right) \prod_{n,L} a_{nL}^{\dagger} \times \\
&\qquad \exp\left(-\sum_n a_n^{\dagger} P_R \left(\sum_{\lambda} u_n(\lambda) u_m^{\dagger}(\lambda)\right) a_m\right). \tag{2.17}
\end{aligned}$$

Note that in the exponential measure factor, the variables a_n are projected into the positive energy modes and the variables $a_n^{\dagger}$ are projected into the right-handed modes, respectively. This is why the product of the two projection operators appears in this measure factor. As we will see later, in the case of a Dirac fermion, this projected measure factor turns into the fermion action defined by the overlap Dirac operator (its chiral part, $P_R D$).

This structure of the projection in the Fock space due to (the integration over) the saturated variables by the vacuum wave functions is a basic

feature of the chiral structure in the overlap formalism. We will see that this structure will play an essential role in the following discussions about the Ginsparg-Wilson relation and associated chiral symmetry.

The measure of the functional integral may be written in terms of the diagonal basis as

$$\prod_n da_n da_n^\dagger = \prod_{\lambda>0} da_u(\lambda) \prod_{\lambda<0} da_v(\lambda) \prod_R da_{nR}^\dagger \prod_L da_{nL}^\dagger. \tag{2.18}$$

The Jacobian factor associated with this change of variable is unity by Eq. (2.2). Then, performing the integration of the saturated variables explicitly, we obtain

$$
\begin{aligned}
\langle\, v\,|\, L\,\rangle &= \int \prod_{\lambda>0} da_u(\lambda) \prod_{nR} da_{nR}^\dagger \exp\left(-\sum_{nR,\lambda>0} a_{nR}^\dagger \cdot u_{nR}(\lambda) \cdot a_u(\lambda) \right) \\
&= \det_{nR,\lambda>0} \{u_{nR}(\lambda)\}.
\end{aligned} \tag{2.19}
$$

Noting the relation which follows from Eq. (2.2),

$$\det_{\lambda>0,nR} \{u_{nR}(\lambda)\} = \frac{1}{\det_{\lambda'<0,L} \{v_{nL}(\lambda')\}} = \det_{\lambda<0,nL} \left\{ v_{nL}^\dagger(\lambda) \right\}, \tag{2.20}$$

we see that this result is identical to Eq. (2.11).

3   From overlap formula to action formalism

Next we consider the partition function of a massless Dirac fermion. In the overlap formalism, it is given by the product of the chiral determinants for right- and left-handed Weyl degrees of freedom. Using chiral conjugation, it may be written as

$$Z_F = \langle\, L\,|\, v\,\rangle \left(\langle\, L\,|\, v\,\rangle \right)^* = \langle\, u\,|\, R\,\rangle \langle\, v\,|\, L\,\rangle. \tag{3.1}$$

This partition function may be written as an overlap between two vacuum states $|\, V\,\rangle = |\, v\,\rangle \otimes |\, u\,\rangle$ and $|\, 5\,\rangle = |\, L\,\rangle \otimes |\, R\,\rangle$ in the product space, which can be regarded as the Dirac vacua of the Hamiltonians [9, 28] given by

$$
\begin{aligned}
\mathcal{H}_D &= \sum_{nm} \hat{a}_n^\dagger H_{nm} \hat{a}_m - \sum_{nm} \hat{b}_n^\dagger H_{nm} \hat{b}_m, & (3.2) \\
\mathcal{H}_{D5} &= \sum_n \hat{a}_n^\dagger \gamma_5 \hat{a}_n - \sum_n \hat{b}_n^\dagger \gamma_5 \hat{b}_n. & (3.3)
\end{aligned}
$$

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$(\hat{b}_n, \hat{b}_n^\dagger)$ is another set of the canonical variables. The partition function of the massless Dirac fermion Eq. (3.1) then may be rewritten in terms of the functional integral over the grassman representatives of the canonical variables $(\hat{a}_n, \hat{a}_n^\dagger)$ and $(\hat{b}_n, \hat{b}_n^\dagger)$:

$$
\begin{aligned}
Z_F &= \int \prod_n da_n da_n^\dagger \prod_n db_n db_n^\dagger \; \exp\left(-\sum_n a_n^\dagger a_n - \sum_n b_n^\dagger b_n \right) \times \\
&\quad \prod_{\lambda<0} \left(\sum_n v_n(\lambda)^\dagger a_n \right) \prod_{\lambda>0} \left(\sum_n u_n(\lambda)^\dagger b_n \right) \prod_n a_{nL}^\dagger \prod_n b_{nR}^\dagger .
\end{aligned}
$$
(3.4)

Due to the projection by the vacuum wave functions, the exponential measure factor may be written as

$$
\sum_n \left(a_n^\dagger a_n + b_n^\dagger b_n \right) \Longrightarrow
$$

$$
\sum_{nm} \left\{ a_n^\dagger P_R \left(\sum_{\lambda>0} u_n(\lambda) u_m^\dagger(\lambda) \right) a_m + b_n^\dagger P_L \left(\sum_{\lambda<0} v_n(\lambda) v_m^\dagger(\lambda) \right) b_m \right\}.
$$
(3.5)

Now we introduce Dirac field variables ψ_n and $\bar{\psi}_n$ defined by

$$
\begin{aligned}
\psi_n &= \frac{1}{\sqrt{2}} \sum_m \left\{ \left(\sum_{\lambda>0} u_n(\lambda) u_m^\dagger(\lambda) \right) a_m + \left(\sum_{\lambda<0} v_n(\lambda) v_m^\dagger(\lambda) \right) b_m \right\} \\
&= \frac{1}{\sqrt{2}} \sum_m \left\{ \frac{1}{2}\left(1 + \frac{H}{\sqrt{H^2}} \right)_{nm} a_m + \frac{1}{2}\left(1 - \frac{H}{\sqrt{H^2}} \right)_{nm} b_m \right\},
\end{aligned}
$$
(3.6)

$$
\begin{aligned}
\bar{\psi}_n &= \frac{1}{\sqrt{2}} \left\{ a_{nR}^\dagger + b_{nL}^\dagger \right\} \\
&= \frac{1}{\sqrt{2}} \left\{ a_n^\dagger \left(\frac{1+\gamma_5}{2} \right) + b_n^\dagger \left(\frac{1-\gamma_5}{2} \right) \right\}.
\end{aligned}
$$
(3.7)

Then it turns out that the exponential weight factor of the grassman integration gives the action of the Dirac field ψ and $\bar{\psi}$ defined by the overlap Dirac operator [9] as

$$
\sum_{nm} \left\{ a_n^\dagger P_R \left(\sum_{\lambda>0} u_n(\lambda) u_m^\dagger(\lambda) \right) a_m + b_n^\dagger P_L \left(\sum_{\lambda<0} v_n(\lambda) v_m^\dagger(\lambda) \right) b_m \right\}
$$

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$$= \sum_{nm} \bar{\psi}_n \left\{ \left(\frac{1+\gamma_5}{2} \right) \left(1 + \frac{H}{\sqrt{H^2}} \right) + \left(\frac{1-\gamma_5}{2} \right) \left(1 - \frac{H}{\sqrt{H^2}} \right) \right\} \psi_m$$

$$= \sum_{nm} \bar{\psi}_n \left(1 + \gamma_5 \frac{H}{\sqrt{H^2}} \right)_{nm} \psi_m. \tag{3.8}$$

By integrating out the variables $a_{nL}^\dagger$, $b_{nR}^\dagger$, $\sum_n v_n(\lambda)^\dagger a_n$ and $\sum_n u_n(\lambda)^\dagger b_n$ saturated by the vacuum wave functions, the partition function can be written in the form

$$Z_F = \int \prod_n d\psi_n d\bar{\psi}_n \, \exp\left(-S_F\right) = \det D, \tag{3.9}$$

$$S_F = \sum_{nm} \bar{\psi}_n D_{nm} \psi_m, \quad D = \left(1 + \gamma_5 \frac{H}{\sqrt{H^2}} \right). \tag{3.10}$$

The Jacobian factor associated with the change of variables of Eqs. (3.6) and (3.7) is also unity by Eq. (2.2). Thus Eqs. (3.6) and (3.7) give the explicit relation between the canonical variables of the auxiliary fermion system in the overlap formalism and the Dirac field variables described by the overlap Dirac operator in the action formalism.

4 Chiral symmetry of Hamiltonian and chiral symmetry of action

Next we discuss the symmetry of the quantum fermion system described by the Hamiltonians Eqs. (3.2) and (3.3) and its relation to the symmetry of the action under the transformation of the type given by Lüscher [21]. As discussed by Narayanan and Neuberger [1], these Hamiltonians possess symmetry under the independent rotations of phases of the two set of the canonical variables $(\hat{a}_n, \hat{a}_n^\dagger)$ and $(\hat{b}_n, \hat{b}_n^\dagger)$:

$$\delta \hat{a}_n = \alpha \, \hat{a}_n, \quad \delta \hat{a}_n^\dagger = -\alpha \, \hat{a}_n^\dagger, \tag{4.1}$$

and

$$\delta \hat{b}_n = \beta \, \hat{b}_n, \quad \delta \hat{b}_n^\dagger = -\beta \, \hat{b}_n^\dagger. \tag{4.2}$$

Recalling the relation between the Dirac field variables ψ_n and $\bar{\psi}_n$ and the grassman representatives of the canonical variables $(a_n, a_n^\dagger)$ and $(b_n, b_n^\dagger)$ given by Eqs (3.6) and (3.7), let us first consider the vector transformation with $\alpha = \beta$:

$$\delta \hat{a}_n = \alpha \, \hat{a}_n, \quad \delta \hat{b}_n = \alpha \, \hat{b}_n. \tag{4.3}$$

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As we can easily see, this transformation induces the vector transformation for ψ and $\bar{\psi}$ as

$$\delta\psi_n = \alpha\,\psi_n, \quad \delta\bar{\psi}_n = -\alpha\,\bar{\psi}_n. \tag{4.4}$$

On the other hand, we can see that the transformation with $\alpha = -\beta$,

$$\delta\hat{a}_n = \alpha\,\hat{a}_n, \quad \delta\hat{b}_n = -\alpha\,\hat{b}_n, \tag{4.5}$$

induces the transformation for ψ and $\bar{\psi}$ as

$$
\begin{aligned}
\delta\psi_n &= \alpha\left\{\sum_\lambda\left(u_n(\lambda)u_m(\lambda)^\dagger\right) - \sum_\lambda\left(v_n(\lambda)v_m(\lambda)^\dagger\right)\right\}\psi_m \\
&= \alpha\left(\frac{H}{\sqrt{H^2}}\right)_{nm}\psi_m \\
&= -\alpha\,\gamma_5\left(1 - aD\right)_{nm}\psi_m, \\
\\
\delta\bar{\psi}_n &= -\alpha\,\bar{\psi}_n\gamma_5.
\end{aligned}
\tag{4.6}
$$

The invariance of the action S_F under the transformation Eq. (4.6) can be easily checked using the Ginsparg-Wilson relation. In this respect, we notice the following fact: for a chiral transformation of the type given by Lüscher to lead the exact symmetry of the action by the Ginsparg-Wilson relation, the weights of the Dirac operator in the transformation for ψ and $\bar{\psi}$ should sum up to unity, but otherwise may be arbitrary. That is, the following transformation with a parameter x also leaves the action invariant:

$$\delta\psi_n = \gamma_5\left(1 - xaD\right)_{nm}\psi_m, \quad \delta\bar{\psi}_n = \bar{\psi}_m\left(1 - (1-x)aD\right)_{mn}\gamma_5, \tag{4.7}$$

$$
\begin{aligned}
\delta S_F &= a^4\sum\bar{\psi}_n\left\{D_{nl}\gamma_5\left(1 - xaD\right)_{lm} + \left(1 - (1-x)aD\right)_{nl}\gamma_5 D_{lm}\right\}\psi_m \\
&= a^4\sum\bar{\psi}_n\left\{D_{nm}\gamma_5 + \gamma_5 D_{nm} - aD_{nl}\gamma_5 D_{lm}\right\}\psi_m = 0.
\end{aligned}
\tag{4.8}
$$

We can also see by considering "local" chiral transformations following the Neother's procedure that all such chiral transformations lead to the same axial vector current [34]. The associated axial Ward-Takahashi identities are physically equivalent. As we have seen, the symmetry of the quantum fermion system in the overlap formalism induces an exact chiral symmetry of the action under one of such chiral transformations defined by Eq. (4.6). This variant of Lüscher's chiral transformation was considered by Neidermeyer, Hasenfratz and Lüscher in more general context where only the Ginsparg-Wilson relation and locality of the Dirac operator are assumed [29].

This invariance of the action S_F can be traced to the invariance of the measure factor

$$\exp\left(-\sum_n \left\{a_n^\dagger a_n + b_n^\dagger b_n\right\}\right) \tag{4.9}$$

under the transformation Eq. (4.5).

5   Anomalous Axial Ward-Takahashi identity in the overlap formalism

Let us next consider from the point of view of the overlap formalism the derivation of the (anomalous) axial Ward-Takahashi identity associated with the chiral transformation Eq. (4.5), which is equivalent to the transformation Eq. (4.6).

The infinitesimal transformation with the local parameter α_n is generated by

$$\hat{G}_\alpha = -\sum_n \left(\hat{a}_n^\dagger \alpha_n \hat{a}_n - \hat{b}_n^\dagger \alpha_n \hat{b}_n\right), \tag{5.1}$$

as

$$\delta|V\rangle = \hat{G}_\alpha|V\rangle, \qquad \delta|5\rangle = \hat{G}_\alpha|5\rangle = 0. \tag{5.2}$$

The invariance of $|5\rangle$ under the axial rotation can be easily understood from the fact that $\mathcal{H}_5$ does not depend on the gauge field and $|5\rangle$ always consists of the equal number of $a_{nL}^\dagger$ and $b_{nR}^\dagger$.

As considered by Narayanan and Neuberger [1, 27] and by Randjbar-Daemi and Strathdee [30], the variation of $|V\rangle$ may be divided into two parts which are parallel and orthogonal to the original vacuum vector:

$$\delta|V\rangle = (\delta|V\rangle)_\perp + |V\rangle\langle V|\hat{G}_\alpha|V\rangle. \tag{5.3}$$

On the other hand, the axial rotation may be regarded to act on the Hamiltonian $\mathcal{H}_D$. This action induces the infinitesimal $U(1)$ axial vector field $B_{n\mu} = \partial_\mu \alpha_n$ in the Hamiltonian. The state vector of the Dirac vacuum is subject to the Wigner-Brillouin phase convention, but its variation does not contribute to the orthogonal variation of the state vector. Therefore we have

$$(\delta|V\rangle)_\perp = \left(\sum_n \partial_\mu \alpha_n \frac{\delta}{\delta B_{n\mu}}|V\rangle(B_\mu)\Big|_{B_\mu=0}\right)_\perp. \tag{5.4}$$

Combining these results, we may write

$$\delta|V\rangle = \left(\sum_n \partial_\mu \alpha_n \frac{\delta}{\delta B_{n\mu}}|V\rangle (B_\mu)\Big|_{B_\mu=0}\right)_\perp + |V\rangle\langle V|\hat{G}_\alpha|V\rangle. \tag{5.5}$$

According to [1, 30, 27], the first term defines the divergence of the axial (covariant) current and the second term gives the associated axial (covariant abelian) anomaly.[3] As we will see, this term is actually identical to the axial anomaly induced from the functional measure in the action formulation [21].

In the overlap formalism, the axial Ward-Takahashi identity follows from a trivial relation,

$$\delta\langle 5|\mathcal{O}|V\rangle = 0. \tag{5.9}$$

It reads

$$\langle 5|\mathcal{O}\left(\partial_\mu \alpha_n \frac{\delta}{\delta B_{n\mu}}|V\rangle (B_\mu)\Big|_{B_\mu=0}\right)_\perp |V\rangle$$
$$+\langle 5|\mathcal{O}|V\rangle\langle V|\hat{G}_\alpha|V\rangle + \langle 5|\left[\hat{G}_\alpha, \mathcal{O}\right]|V\rangle = 0 \tag{5.10}$$

This identity is equivalent to the axial Ward-Takahashi identity in the action formulation when the operator $\mathcal{O}$ consists of only $\hat{a}_u$, $\hat{b}_v$, $\hat{a}_R$ and $\hat{b}_L$, which correspond to ψ and $\bar{\psi}$.

We will evaluate the second term of the axial anomaly and the first term of the axial current in terms of the grassman-number-integral representation given by Eq. (3.4).

$$Z_F = \int \prod_n da_n da_n^\dagger \prod_n db_n db_n^\dagger \, \Psi_v^\dagger(a_n) \, \Psi_u^\dagger(b_n) \prod_n a_{nL}^\dagger \prod_n b_{nR}^\dagger \times$$
$$\exp\left(-\sum_n a_n^\dagger a_n - \sum_n b_n^\dagger b_n\right). \tag{5.11}$$

---

[3] Using the Hamiltonian perturbation theory, we can obtain the expression of the orthogonal variation of the Dirac vacuum [27] as

$$(\delta|V\rangle)_\perp = \frac{1}{\mathcal{H}_D - E_0}\left(\langle V|\delta\mathcal{H}_D|V\rangle - \delta\mathcal{H}_D\right)|V\rangle. \tag{5.6}$$

$$\delta\mathcal{H}_D = \sum_{n\mu} \partial_\mu \alpha_n \left\{\hat{a} W_{n\mu} a - \hat{b} W_{n\mu} b\right\} \tag{5.7}$$

$$W_{n\mu}(s,t) = \gamma_5 \frac{1}{2}(\gamma_\mu - 1)\delta_{sn}\delta_{n+\hat{\mu},t} U_{n\mu} + \gamma_5 \frac{1}{2}(\gamma_\mu + 1)\delta_{s,n+\hat{\mu}}\delta_{nt} U_{n+\hat{\mu},\mu}. \tag{5.8}$$

In this representation, the axial Ward-Takahashi identity follows from the change of variables along the chiral transformation in the grassman-number-integral:

$$\delta Z_F = 0, \tag{5.12}$$

$$\delta a_n = \alpha_n\, a_n, \quad \delta b_n = -\alpha_n\, b_n. \tag{5.13}$$

The axial anomaly can be evaluated from the parallel variation of the vacuum wave functions $|V\rangle$ in the grassman number representation:

$$\Psi_v^\dagger(a_n)\,\Psi_u^\dagger(b_n) = \prod_{\lambda<0}\left(\sum_n v_n(\lambda)^\dagger a_n\right)\prod_{\lambda>0}\left(\sum_n u_n(\lambda)^\dagger b_n\right). \tag{5.14}$$

The variation of the creation operators which consist the Dirac vacuum can be written as

$$\delta\left(\sum_n v_n(\lambda)^\dagger a_n\right) = \sum_{nm} v_n(\lambda)^\dagger \alpha_n\left(\sum_{\lambda'>0} u_n(\lambda')u_m(\lambda')^\dagger\right)a_m$$
$$+ \sum_{nm} v_n(\lambda)^\dagger \alpha_n\left(\sum_{\lambda'<0} v_n(\lambda')v_m(\lambda')^\dagger\right)a_m. \tag{5.15}$$

The first term of the r.h.s. consists of the positive energy eigenstates and induces the variation orthogonal to the original vacuum state. On the other hand, the second term of the r.h.s. consists of the negative energy eigenstates and the only single term with $\lambda' = \lambda$ can contribute so that it recovers the vacuum vector. Therefore we obtain

$$\delta\Psi_v^\dagger(a_n) = \left(\delta\Psi_v^\dagger(a_n)\right)_\perp + \left\{\sum_{\lambda<0} v_n(\lambda)^\dagger \alpha_n v_n(\lambda)\right\}\Psi_v^\dagger(a_n), \tag{5.16}$$

where

$$\left(\delta\Psi_v^\dagger(a_n)\right)_\perp = \sum_{nm} v_n(\lambda)^\dagger \alpha_n\left(\sum_{\lambda>0} u_n(\lambda)u_m(\lambda)^\dagger\right)a_n \prod_{\lambda<0/\lambda'}\left(\sum_n v_n(\lambda)^\dagger a_n\right). \tag{5.17}$$

Here $\prod_{\lambda<0/\lambda'}$ stands for the product over negative eigenvalues except that the element of λ' is removed after it is moved all the way to the most left

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term, taking into account the ordering. Similar consideration applies to the wave function $\Psi_u(b_n^\dagger)$. Then the parallel variations give rise to the factor

$$\left\{ \sum_{\lambda<0} v_n(\lambda)^\dagger \alpha_n v_n(\lambda) - \sum_{\lambda>0} u_n(\lambda)^\dagger \alpha_n u_n(\lambda) \right\} = -\mathrm{Tr}\,\alpha_n \left(\frac{H}{\sqrt{H^2}} \right)_{nn}$$
$$= -\mathrm{Tr}\,\alpha_n \gamma_5 D_{nn}, \quad (5.18)$$

which is identical to the anomaly coming from the variation of the functional measure in the action formulation.

The axial vector current may be evaluated in a similar manner from the orthogonal variation of the vacuum wave functions. In this respect, we notice the following simplification due to the projection by the vacuum wave functions: the exponential measure factor in the partition function Z_F can be rewritten as Eq. (3.5),

$$\sum_{nm} \left\{ a_n^\dagger P_R \left(\sum_{\lambda>0} u_n(\lambda) u_m^\dagger(\lambda) \right) a_m + b_n^\dagger P_L \left(\sum_{\lambda<0} v_n(\lambda) v_m^\dagger(\lambda) \right) b_m \right\},$$

in which no negative energy modes of a_n and no positive energy modes of b_n appear. As we have seen, the orthogonal variation of the vacuum implies the creation of holes of very these modes. Since there is not any other supply of these modes in the grassman number integral, this contribution from the orthogonal variation of the vacuum must vanish. Instead, the variation of the *projected* measure factor gives rise to the axial vector current. The part from the variables $(a_n, a_n^\dagger)$ is evaluated as follows:

$$P_R \left\{ \left(\sum_{\lambda>0} u_n(\lambda) u_m^\dagger(\lambda) \right) \alpha_m - \alpha_n \left(\sum_{\lambda>0} u_n(\lambda) u_m^\dagger(\lambda) \right) \right\}$$
$$= P_R \left\{ \left(\frac{H}{\sqrt{H^2}} \right)_{nm} \alpha_m - \alpha_n \left(\frac{H}{\sqrt{H^2}} \right)_{nm} \right\}$$
$$= P_R \left\{ D_{nm} \alpha_m - \alpha_n D_{nm} \right\}. \quad (5.19)$$

Introducing the kernel of the vector current by [4]

$$D(n,m)\alpha_m - \alpha(n)D(n,m) = \partial_\mu \, \alpha_l K_{l\mu}(n,m), \quad (5.20)$$

and taking into account of the projection by the vacuum wave functions again, the current can be written as

$$- J_{l\mu}^{L-} = -\sum_{nm} \left\{ a_n^\dagger P_R K_{l\mu}(n,m) P_- a_m \right\}. \quad (5.21)$$

---

[4]As to the explicit formula of the kernel, see [34] .

where

$$P_{\pm} = \frac{1}{2} \left(1 \mp \frac{H}{\sqrt{H^2}} \right) = \frac{1 \pm \gamma_5 (1 - aD)}{2}. \tag{5.22}$$

Combining the part from the variables $(b_n, b_n^\dagger)$, the axial current is written as

$$- J_{l\mu}^{L-} + J_{l\mu}^{R+} = \sum_{nm} \left\{ -a_n^\dagger P_R K_{l\mu}(n,m) P_- a_m + b_n^\dagger P_L K_{l\mu}(n,m) P_+ b_m \right\}. \tag{5.23}$$

This is an expression of the axial current of Eqs. (5.4) and (5.6). In terms of the Dirac fields ψ_n and $\bar{\psi}_n$, we may write it as

$$- J_{l\mu}^{L-} + J_{l\mu}^{R+} = \sum_{nm} \left\{ -\bar{\psi}_n P_R K_{l\mu}(n,m) P_- \psi_m + \bar{\psi}_n P_L K_{l\mu}(n,m) P_+ \psi_m \right\}. \tag{5.24}$$

6 Covariant gauge current and gauge anomaly in covariant form

In this section, we consider the covariant gauge current and the associated gauge anomaly in the covariant form discussed by Narayanan and Neuberger [11], by Randjbar-Daemi and Strathdee [30] and by Neuberger[27, 28] from the point of view of the grassman-number-integral representation.

Just like the axial Ward-Takahashi identity associated with the chiral transformation Eqs. (5.12) and (6.1), we may consider an identity associated with the following local transformation:

$$\delta a_{n\alpha} = (\omega_n)_\alpha^\gamma \, a_{n\gamma}, \quad \delta a_n^{\dagger\beta} = - a_n^{\dagger\gamma} \, (\omega_n)_\gamma^\beta, \tag{6.1}$$

where $\omega = \omega^a T^a$ and T^a is the generators of the gauge group.[5] The identity follows from

$$\delta \langle \, v \, | \, L \, \rangle = 0, \tag{6.2}$$

where $\langle \, v \, | \, L \, \rangle$ is given by Eq. (2.17). In the same way as the previous section, we can obtain its explicit form in the grassman number representation.

The anomaly term is simply obtained as

$$\sum_n \omega_n^a \left(-a \operatorname{tr} T^a \gamma_5 D_{nn} \right). \tag{6.3}$$

---

[5] We assume a certain normalization of the generators in abelian subgroups. We use the Greek letters for the group indices in this section.

The trace is taken over the spinor and gauge group indices.

The expression of the covariant gauge current is given by

$$\{J_{l\mu}\}_{\beta}^{\alpha} = \sum_{nm} a_n^{\dagger} P_R \{K_{l\mu}\}_{\beta}^{\alpha} P_-(n,m) a_m, \tag{6.4}$$

where the kernel of the current is defined from the overlap Dirac operator by

$$K_{l\mu}(n,m)_{\beta}^{\alpha} = \sum_{\gamma} U_{\mu}(l)_{\beta}^{\gamma} \frac{\delta}{\delta U_{\mu}(l)_{\alpha}^{\gamma}} D[U]. \tag{6.5}$$

Its explicit form can be obtained using the integral representation of the square root of H^2 in the overlap Dirac operator [34]:

$$aK_{l\mu}(n,m)_{\beta}^{\alpha} = \gamma_5 \left\{ \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{(t^2 + H^2)} \left(t^2 (W_{l\mu})_{\beta}^{\alpha} - H (W_{l\mu})_{\beta}^{\alpha} H \right) \frac{1}{(t^2 + H^2)} \right\}_{nm}, \tag{6.6}$$

$$\gamma_5 \left\{ W_{l\mu}(n,m)_{\sigma}^{\rho} \right\}_{\beta}^{\alpha} \tag{6.7}$$

$$\equiv \sum_{\gamma} U_{\mu}(l)_{\beta}^{\gamma} \frac{\delta}{\delta U_{\mu}(l)_{\alpha}^{\gamma}} D_W[U]_{\sigma}^{\rho}$$

$$= \frac{1}{2} (\gamma_{\mu} - 1) \delta_{ln} \delta_{n+\hat{\mu},m} \delta_{\sigma}^{\alpha} \{U_{l\mu}\}_{\beta}^{\rho} + \frac{1}{2} (\gamma_{\mu} + 1) \delta_{lm} \delta_{n,m+\hat{\mu}} \left\{ U_{l\mu}^{-1} \right\}_{\sigma}^{\alpha} \delta_{\beta}^{\rho}. \tag{6.8}$$

Then the explicit form of the identity reads

$$\left\langle \{D_{\mu}^* J_{l\mu}\}^a \right\rangle = -a \operatorname{tr} \left\{ T^a \gamma_5 D(l,l) \right\}. \tag{6.9}$$

The expectation value of the current in the l.h.s. stands for the grassman-number-integration over the variables $(a_n, a_n^{\dagger})$ as in Eq. (2.17). Explicitly, it is given by

$$\langle J_{l\mu} \rangle = -\operatorname{Tr} \left\{ P_R K_{l\mu} P_- D^{-1} \right\}. \tag{6.10}$$

(See Eq. (7.2)) The trace in the r.h.s. is taken over the lattice, spinor and gauge group indices. The covariant derivative is defined as

$$\{D_{\mu}^* J_{l\mu}\}^a \equiv \{T^a\}_{\alpha}^{\beta} \left(\{J_{l\mu}\}_{\beta}^{\alpha} - \left\{ U_{l-\hat{\mu},\mu}^{-1} \right\}_{\beta}^{\gamma} \{J_{l-\hat{\mu},\mu}\}_{\gamma}^{\sigma} \{U_{l-\hat{\mu},\mu}\}_{\sigma}^{\alpha} \right)$$

$$= \operatorname{tr} \left\{ T^a \left(J_{l\mu} - U_{l-\hat{\mu},\mu}^{-1} J_{l-\hat{\mu},\mu} U_{l-\hat{\mu},\mu} \right) \right\}. \tag{6.11}$$

Eq. (6.4) gives an expression of the covariant gauge current discussed in [11, 30, 27, 28]. The covariance follows from its definition. The associated

anomaly, which is given by the r.h.s of Eq. (6.9) or Eq. (6.3), is also covariant. This is an extension of the chiral anomaly obtained by Lüscher [21] to the case of the gauge symmetry.

In fact, the covariant gauge current of the type Eq. (6.4) and the identity Eq. (6.9) can be obtained in a more general class of the fermion systems based on the Dirac operators which satisfy the Ginsparg-Wilson relation. The local transformations,

$$\delta \psi = P_- \, \omega \, P_- \, \psi, \quad \delta \bar{\psi} = -\bar{\psi} \, P_R \, \omega \qquad (6.12)$$

give rise to the identity of the form Eq. (6.9), where the current is given by Eq. (6.4) with the field variables a_n and $a_n^\dagger$ replaced by ψ and $\bar{\psi}$, respectively. This is easily understood from the relation between $\psi, \bar{\psi}$ and $a_n, a_n^\dagger$ given by Eqs. (3.6) and (3.7), in the case of the overlap formalism and the overlap Dirac operator in discussion.

7   Discussion – Fermion correlation function in the overlap formalism and the Ginsparg-Wilson relation –

Finally, we discuss how the Ginsparg-Wilson relation comes out from the point of view of the overlap formalism. The Ginsparg-Wilson relation itself was derived originally through the idea of the block-spin transformation. On the other hand, the Dirac operator derived by Neuberger is based on the overlap formalism of the chiral determinant and may appear at first sight to have nothing to do with such a relation based on the renormalization group method. We will see how the structure of the projection due to the Dirac vacua wave functions leads to the Ginsparg-Wilson relation.

The fermion correlation function in the overlap formalism has also a simple relation to the fermion propagator in the action formulation.

$$\langle \, 5 \, | \left\{ \hat{a}_n \hat{a}_m^\dagger + \hat{b}_n \hat{b}_m^\dagger \right\} | \, V \, \rangle / \langle \, 5 \, | \, V \, \rangle = \left(D^{-1} \right)_{nm} . \qquad (7.1)$$

In fact, the l.h.s. is evaluated as follows:

$$\begin{aligned}
&(\text{l.h.s.}) \\
=\; & \frac{1}{Z_F} \int \prod_n da_n da_n^\dagger \prod_n db_n db_n^\dagger \exp \left(- \sum_n \left(a_n^\dagger a_n + b_n^\dagger b_n \right) \right) \times \\
& \Psi_v^\dagger(a_n) \Psi_L(a_n^\dagger) \Psi_u^\dagger(b_n) \Psi_R(b_n^\dagger) \left\{ \hat{a}_n \hat{a}_m^\dagger + \hat{b}_n \hat{b}_m^\dagger \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{Z_F} \int \prod_n d\psi_n d\bar{\psi}_n \, \exp\left(-\bar{\psi}_n D_{nm}\psi_m\right) \left\{\psi_{n-}\bar{\psi}_{mL} + \psi_{n+}\bar{\psi}_{mR}\right\} \\
&= \left(D^{-1}\right)_{nm}.
\end{aligned}
\tag{7.2}
$$

Then the Ginsparg-Wilson relation for D^{-1} can be traced to the relation of the fermion correlation function in the overlap formalism as follows (for single Weyl fermion):

$$
\begin{aligned}
P_R \delta_{nm} &= \frac{1}{Z_F} \int \prod_n da_n da_n^\dagger \, \exp\left(-\sum_n a_n^\dagger a_n\right) \times \\
&\quad \Psi_v^\dagger(a_n)\Psi_L(a_n^\dagger) \left\{a_n a_m^\dagger, \gamma_5\right\}.
\end{aligned}
\tag{7.3}
$$

Let us examine how this relation holds true by evaluating the r.h.s. of Eq. (7.3). By the projections due to the vacuum wave functions $\Psi_L(a_n^\dagger)$, the variable $a_n^\dagger$ is projected to the right-handed component. Then, because of the anti-commutator with γ_5 matrix, it turns out that a_n is also projected to the right-handed component. The r.h.s. of this equation can be written as

$$
\begin{aligned}
(r.h.s.) &= 2 \int \prod da_{nR} da_{nL} da_{nR}^\dagger \, \exp\left(-\sum_n a_{nR}^\dagger a_{nR}\right) \times \\
&\quad \Psi_v^\dagger(a_n) \, a_{nR} \, a_{mR}^\dagger.
\end{aligned}
\tag{7.4}
$$

Here we have used the chiral basis also for a_n. Since the exponential measure factor does not contain the variables a_{nL}, all of them must be supplied from the vacuum wave function of $|V\rangle$. Then this vacuum wave function cannot contribute in the evaluation of the correlations of $\left\{a_{nR}a_{mR}^\dagger\right\}$ and it is determined entirely by that *gaussian* measure factor:

$$
\begin{aligned}
(r.h.s.) &\simeq \int \prod da_{nR} da_{nR}^\dagger \, \exp\left(-\sum_n a_{nR}^\dagger a_{nR}\right) a_{nR} a_{mR}^\dagger \\
&\simeq \delta_{nm}.
\end{aligned}
\tag{7.5}
$$

Thus we can see that the Ginsparg-Wilson relation holds in the overlap formalism entirely due to the projection by the wave function of the vacuum $|5\rangle$. The detail structure of the vacuum $|V\rangle$ is irrelevant in this respect. This result is related to the fact that the Ginsparg-Wilson relation itself does not determine the physical properties of the Dirac operator such as the structure of the pole of the propagator. This depends on the choice of the

Hamiltonian H. This point was emphasized also by Chiu, Wang and Zenkin [35].

We also notice the fact that the strict locality of the right-hand-side of the Ginsparg-Wilson relation, which is realized in the overlap formalism, is related to the choice of $H_5 = \gamma_5$. This is the direct consequence of letting the mass of the domain-wall fermion infinity in its positive region: $+m_0 = \infty$ [36, 8].

The grassman number representation of the overlap formula, which we have utilized extensively in this paper, may also be useful, for example, in deriving the Schwinger-Dyson equations in the overlap formalism.

Acknowledgments

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